MATH 245 S20, Exam 3 Solutions

1. Let $S = \{a, b, c, d, e\}$. Find all partitions of S into exactly three parts, such that a, b are in different parts. Warning: missing, duplicate, extra partitions will all cost points.

   Ten partitions into three parts have one part of size 3: $\cancel{\{\{a,b,c\},\{d\},\{e\}\}}$, $\cancel{\{\{a,b,d\},\{c\},\{e\}\}}$, $\cancel{\{\{a,b,e\},\{c\},\{d\}\}}$, $\{\{a,c,d\},\{b\},\{e\}\}$, $\{\{a,c,e\},\{b\},\{d\}\}$, $\{\{a,d,e\},\{b\},\{c\}\}$, $\{\{b,c,d\},\{a\},\{e\}\}$, $\{\{b,c,e\},\{a\},\{d\}\}$, $\{\{b,d,e\},\{a\},\{c\}\}$, $\{\{c,d,e\},\{a\},\{b\}\}$; however three of them put a, b into the same part, which leaves seven.

   There are fifteen partitions into three parts, with two parts of size 2: $\cancel{\{\{a,b\},\{c,d\},\{e\}\}}$, $\cancel{\{\{a,b\},\{c,e\},\{d\}\}}$, $\cancel{\{\{a,b\},\{d,e\},\{c\}\}}$, $\{\{a,c\},\{b,d\},\{e\}\}$, $\{\{a,c\},\{b,e\},\{d\}\}$, $\{\{a,c\},\{d,e\},\{b\}\}$, $\{\{a,d\},\{b,c\},\{e\}\}$, $\{\{a,d\},\{b,e\},\{c\}\}$, $\{\{a,d\},\{c,e\},\{b\}\}$, $\{\{a,e\},\{b,c\},\{d\}\}$, $\{\{a,e\},\{b,d\},\{c\}\}$, $\{\{a,e\},\{c,d\},\{b\}\}$, $\{\{b,c\},\{d,e\},\{a\}\}$, $\{\{b,d\},\{c,e\},\{a\}\}$, $\{\{b,e\},\{c,d\},\{a\}\}$; however three of them put a, b into the same part, which leaves twelve.

   Altogether there are $12 + 7 = 19$ partitions.

   This problem tests the subtleties of the definition of "partition", from section 9.1.

2. Prove or disprove: For all sets A, B, C, D, we have $(A\Delta B) \times (C\Delta D) = (A \times C)\Delta(B \times D)$.

   The statement is false. To disprove requires a counterexample. Many, many correct solutions are possible. Grading will be based on finding a correct counterexample, and (primarily) a correct proof that what you have provided is indeed a counterexample.

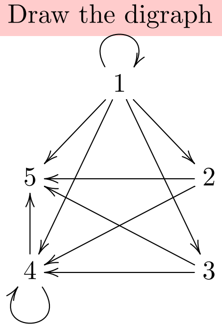
   One correct solution is: Let $A = D = \{x\}, B = C = \emptyset$. Then $x \in A\Delta B$ (since $x \in A$ and $x \notin B$) and $x \in C\Delta D$ (since $x \in D$ and $x \notin C$), so $(x, x) \in (A\Delta B) \times (C\Delta D)$. However, $(x, x) \notin (A \times C)$, since $x \notin C$. Also $(x, x) \notin (B \times D)$, since $x \notin B$. Thus $(x, x) \notin (A \times C)\Delta(B \times D)$.

   This problem tests the definitions of symmetric difference, Cartesian product, set equality, and proof structures for same. It also tests student intuition (should we try to prove or disprove?), gained from exercises like 8.10, 8.19, 9.16, 9.17, 9.18, 9.19.

3. Let $S = \{1, 2, 3, 4, 5\}$. Let R be the relation on S given as

   $$R = \{(a, b) : \exists c \in \mathbb{Z}, a \leq c^2 \leq b\}.$$

   Draw the digraph representing R.

   e.g.: $(2, 3) \notin R$ since $\forall c \in \mathbb{Z}, c^2 \notin [2, 3]$. $(1, 5) \in R$ since $c = 1$ has $c^2 \in [1, 5]$.

   By considering $c = 1$, we have $(1, b) \in R$ for all $b \in S$. By considering $c = 2$, we have $(2, 4), (2, 5), (3, 4), (3, 5), (4, 4), (4, 5)$ all in R. There should be 11 directed edges. Any missing or extra edges will result in lost points.

   This problem tests set-builder notation (section 8.1) and digraph representation of relations (section 10.1).

4. Let S, T be sets, satisfying the property $\forall x \in T,\ x \notin S$. Prove that $S \cap T = \emptyset$.

   We have $\emptyset \subseteq S \cap T$ since $\emptyset$ is a subset of every set. The hard direction is that $S \cap T \subseteq \emptyset$. Let $x \in S \cap T$. Then $x \in S \wedge x \in T$. By simplification twice, $x \in S$ and $x \in T$. We now apply our hypothesis to $x \in T$ conclude that $x \notin S$. But now $x \in S$ and $x \notin S$, which is a contradiction. Hence $(x \in S \cap T) \to (x \in \emptyset)$ is vacuously true.

   This problem tests the proof structures for proofs about sets. It also tests proofs involving the empty set (e.g. vacuous proofs). These skills were developed in exercises like 8.3, 8.14, 8.15, 8.16, 8.17, 8.18. As a little bonus, this exact question was on the fake exam.

5. Let S be a set, and R an antisymmetric relation on S. Prove that R^c is trichotomous.

   Let $x, y \in S$, and suppose that $(x, y) \notin R^c$ and $(y, x) \notin R^c$. Hence, $(x, y) \in R$ and $(y, x) \in R$. Since R is antisymmetric, $x = y$.

   This problem tests the definitions of antisymmetric, trichotomous and set complement, as well as the proof structures for sets and relations. It is similar to exercises 10.11, 10.12, 10.14, 10.18, 10.19, 10.20.

6. Let S be a set, and R a relation on S. Let R' be the symmetric closure of R. Prove that R' is symmetric.

Let $x, y \in S$ be arbitrary, and suppose that $(x, y) \in R' = R \cup R^{-1}$. Hence $(x, y) \in R \vee (x, y) \in R^{-1}$. There are now two cases:

1. Case $(x, y) \in R$. Now, $(y, x) \in R^{-1}$, so by addition $(y, x) \in R \vee (y, x) \in R^{-1}$. Hence $(y, x) \in R \cup R^{-1} = R'$.
2. Case $(x, y) \in R^{-1}$. Now, $(y, x) \in R$, so by addition $(y, x) \in R \vee (y, x) \in R^{-1}$. Hence $(y, x) \in R \cup R^{-1} = R'$.

In both cases, $(y, x) \in R'$.

This problem tests the definitions of symmetric and symmetric closure, as well as the proof structures for sets and relations. It is exercise 10.27, and is also similar to 10.18, 10.26, 10.28.

7. Let R, S, T be sets. Prove that $(R \cap S) \cup (R \cap T) \subseteq R \cap (S \cup T)$. Do not use Thm 8.15.

Let $x \in (R \cap S) \cup (R \cap T)$. Then $x \in (R \cap S) \vee x \in (R \cap T)$. This is logically equivalent to $(x \in R \wedge x \in S) \vee (x \in R \wedge x \in T)$. By Theorem 2.12a (distributivity for propositions), this is equivalent to $x \in R \wedge (x \in S \vee x \in T)$. This is equivalent to $x \in R \wedge x \in S \cup T$, which finally is equivalent to $x \in R \cap (S \cup T)$.

This problem tests the basic definitions of union, intersection, and subset, in a slightly more complicated setting. It is part of exercise 8.23, i.e. theorem 8.15. This problem is all about establishing and using a good proof structure.

8. Prove or disprove that $S = T$, for

$$S = \{x \in \mathbb{Z} : \exists y, z \in \mathbb{Z},\ x = 8y \wedge x = 6z\},$$

$$T = \{x \in \mathbb{Z} : \exists y, z \in \mathbb{Z},\ x = 8y \wedge x = 3z\}.$$

The statement is true; there are two subset relationships to prove.

$S \subseteq T$ (easier): Let $x \in S$. Then there are $y, z \in \mathbb{Z}$ with $x = 8y$ and $x = 6z$. We have $x = 3(2z)$, and $2z \in \mathbb{Z}$, so $x \in T$.

$T \subseteq S$ (harder): Let $x \in T$. Then there are $y, z \in \mathbb{Z}$ with $x = 8y$ and $x = 3z$. Now, $8y = x = 3z$, so $2|3z$. Since 2 is prime, either $2|3$ (which it doesn't) or $2|z$. Since $2|z$, there is some $z' \in \mathbb{Z}$ with $z = 2z'$. Hence, $x = 3z = 6z'$, so $x \in S$.

This problem tests the basic definitions of set equality and set-builder notation, in a slightly more complicated setting involving some basic number theory. It is similar to exercises 8.4, 8.5, 8.6, 8.7.

9. Let A, B, C, D be sets with $|A| = |B|$ and $C \subseteq D$. Prove that $|A \times C| \leq |B \times D|$.

We first prove that $B \times C \subseteq B \times D$: let $(x, y) \in B \times C$. Then $x \in B$ and $y \in C$, but since $C \subseteq D$ in fact $y \in D$, so $(x, y) \in B \times D$.

Now, it is enough to prove that $|A \times C| = |B \times C|$, by Def. 9.15. We will prove this with a pairing, that uses the pairing $a \leftrightarrow b$ between $a \in A$ and $b \in B$ (such a pairing exists because $|A| = |B|$). We build our pairing via $(a, c) \leftrightarrow (b, c)$, where a, b are paired, and we have the same c on both sides.

NOTE: The alternate proof $|A \times C| = |A||C| = |B||C| \leq |B||D| = |B \times D|$ works only for finite sets, and will only receive partial credit.

This problem tests the definitions of cardinality, both for $=$ and for $\leq$, as well as the definition of Cartesian product (section 9.3).

10. We say that set S has an *ascending chain* if there are infinitely many distinct sets $S_1, S_2, S_3, \ldots$ with $S \subseteq S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots$. We say that set S has a *descending chain* if there are infinitely many distinct sets $S_1, S_2, S_3, \ldots$ with $S \supseteq S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$. Consider set S given by $S = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}, x = 2y\} = \{2x : x \in \mathbb{Z}\}$. Prove that S has both an ascending chain and a descending chain.

An ascending chain $S \subseteq S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots$ is given by $S_1 = S \cup \{1\}$, $S_2 = S \cup \{1, 3\}$, $S_3 = S \cup \{1, 3, 5\}$, $\ldots$, $S_k = S \cup \{2i - 1 : 1 \leq i \leq k\}$. (other answers are possible)

A descending chain $S \supseteq S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$ is given by $S_1 = S \setminus \{2\}$, $S_2 = S \setminus \{2, 4\}$, $S_3 = S \setminus \{2, 4, 6\}$, $\ldots$, $S_k = S \setminus \{2i : 1 \leq i \leq k\}$. (other answers are possible)

This problem tests the ability to learn and use new definitions (aka courage). It tests the basic understanding of subset, set equality, and set-builder notation.